

NON-ALTERNATING AND NON-SEPARATING TRANSFORMATIONS
MODULO A FAMILY OF SETS

By
JOHN W. ODLE

A DISSERTATION
SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN THE
UNIVERSITY OF MICHIGAN

Reprinted from DUKE MATHEMATICAL JOURNAL
Vol. 8, Number 2, 1941

Made in the United States of America



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1. **Introduction.** In a previous paper, by E. P. Vance [5],¹ weakly non-separating, weakly non-alternating, weakly completely non-separating, and weakly completely non-alternating transformations have been defined and discussed. It is the purpose of this paper to study some natural generalizations of these four transformations and to obtain Vance's types as one of the special cases. In the above transformations special favor was allotted to sets consisting of but a single point by allowing them to separate, whereas in the generalized transformations which are to be defined here, finite sets, continua, and other types of sets will be permitted to effect separations. This is the basis of the generalization accomplished in this paper. It will be seen that many of the same theorems which Vance obtained will go through in the more general situation also.

In the last two sections of this paper locally non-separating and locally non-alternating transformations, which were also defined by Vance in his paper, will be studied further, and some relationships between 0-regular transformations and locally non-alternating transformations will be established.

In this paper all transformations, $T(A) = B$, are assumed to be single valued and continuous, and the spaces considered are assumed to be compact metric continua.

The four new types of continuous transformations which are to be studied in this paper are defined as follows:

(1) T is called *non-separating modulo $\mathcal{G}$* if, for any point x of B , the set $T^{-1}(x)$ does not separate two points in A unless a subset of $T^{-1}(x)$ which belongs to the collection $\mathcal{G}$ also separates the same two points in A .

(2) T is called *non-alternating modulo $\mathcal{G}$* if, for any two points x and y of B , the set $T^{-1}(x)$ does not separate two points of $T^{-1}(y)$ in A unless a subset of $T^{-1}(x)$ which belongs to the collection $\mathcal{G}$ also separates the same two points in A .

(3) T is called *completely non-separating modulo $\mathcal{G}$* if, for any point x of B and any closed subset M of $T^{-1}(x)$, M does not separate two points in A unless a subset of M which belongs to the collection $\mathcal{G}$ also separates the same two points in A .

Received August 20, 1940; presented to the American Mathematical Society, November 23, 1940. The writer wishes to express his appreciation for the help given by Professor W. L. Ayres of the University of Michigan, who suggested the problem and gave valuable criticism and advice during the preparation of this paper.

¹ Numbers in brackets refer to the bibliography at the end of the paper.

(4) T is called *completely non-alternating modulo $\mathcal{G}$* if, for any two points x and y of B and any closed subset M of $T^{-1}(x)$, M does not separate two points of $T^{-1}(y)$ in A unless a subset of M which belongs to the collection $\mathcal{G}$ also separates the same two points in A . (x and y may be identical here.)

If the null set is taken to be the only set in the collection $\mathcal{G}$, then the preceding definitions reduce to the definitions of non-separating, non-alternating, completely non-separating, and completely non-alternating transformations, which have been studied in considerable detail in previous papers by other authors [7, 9].

If the collection $\mathcal{G}$ is defined to be the totality of all single-point sets of the space A , then the definitions given here reduce to the definitions of weakly non-separating, weakly non-alternating, weakly completely non-separating, and weakly completely non-alternating transformations, which were given by Vance [5] in his recent paper.

Other collections $\mathcal{G}$ which will be mentioned later are the class of continua, the class of finite sets, and the class of sets having a finite number of components. Other meanings are of course possible, and perhaps useful. The beauty of the definitions of the transformations given here lies in their generality and applicability to many special cases.

It should be noted that if $\mathcal{G}$ is the collection of all closed sets, then any continuous transformation satisfies all four of the definitions which we have given above. This follows directly from the definitions and from the fact that $T^{-1}(x)$ is a closed set, for any x . This means, of course, that such a liberal definition for the collection $\mathcal{G}$ is unproductive of interesting results.

Certain relations exist between the transformations here defined, independently of the definition given to the collection $\mathcal{G}$. These relations are: If T is completely non-separating modulo $\mathcal{G}$, then T is completely non-alternating modulo $\mathcal{G}$; if T is completely non-separating modulo $\mathcal{G}$, then T is non-separating modulo $\mathcal{G}$; if T is completely non-alternating modulo $\mathcal{G}$, then T is non-alternating modulo $\mathcal{G}$; if T is non-separating modulo $\mathcal{G}$, then T is non-alternating modulo $\mathcal{G}$. These relations follow directly from the definitions given for the various types of transformations. By means of simple examples easily constructible for any of the specific collections $\mathcal{G}$ which we have considered, it can be seen that these are the only relations of implication which always subsist between these transformations, independently of the definition used for collections $\mathcal{G}$.

2. Some characteristic properties. Vance proved a number of theorems which delineated the properties of the transformations which he defined. Most of these theorems are still valid when stated for the general modulo $\mathcal{G}$ transformations with the collection $\mathcal{G}$ unspecified. The statements of these theorems will now be given, but the proofs will be omitted, as they follow the pattern of Vance's proofs with very slight modification. These theorems hold independently of the definition given to collection $\mathcal{G}$.

THEOREM 2.1. *If the transformation T is non-separating modulo $\mathfrak{G}$, then, for any cut point b of B , the set $T^{-1}(b)$ contains a set K of the collection $\mathfrak{G}$ which separates the space A .*

THEOREM 2.2. *If T is completely non-alternating modulo $\mathfrak{G}$, then every open set contained in any set $T^{-1}(x)$ must contain a set K of the collection $\mathfrak{G}$ which is a cut set of A .*

COROLLARY 2.21. *If T is completely non-separating modulo $\mathfrak{G}$, then every open set contained in any set $T^{-1}(x)$ must contain a cut set of A which is in the collection $\mathfrak{G}$.*

THEOREM 2.3. *If T is completely non-separating modulo $\mathfrak{G}$, and if b is a cut point of B , and if $T^{-1}(b)$ contains an irreducible cut set M between two points of A , then M is a set of the collection $\mathfrak{G}$.*

THEOREM 2.4. *Let A , and therefore B , be locally connected. If T is non-alternating and if, for any cut point b of B , every subset of $T^{-1}(b)$ which is an irreducible cut set of A between two points of A is a set of the collection $\mathfrak{G}$, then T is non-separating modulo $\mathfrak{G}$.*

THEOREM 2.5. *Let A , and therefore B , be locally connected. If T is completely non-alternating and if, for any cut point b of B , every subset of $T^{-1}(b)$ which is an irreducible cut set of A between two points of A is a set of the collection $\mathfrak{G}$, then T is completely non-separating modulo $\mathfrak{G}$.*

3. Product and factor theorems. With the purpose of obtaining a complete analysis of all product and factor theorems involving the following five types of transformations: monotone, non-separating, non-alternating, non-separating modulo $\mathfrak{G}$ and non-alternating modulo $\mathfrak{G}$, where the collection $\mathfrak{G}$ is the totality of all single-point sets, we have set down all possible combinations of these transformations and investigated every one to see if it would lead to a true theorem. As a result of this systematic investigation, it can now be stated definitely that the previously known product and factor theorems, given by Vance [5], Wardwell [7], and Whyburn [9], plus two new ones included in this paper for the first time, are the only true product and factor theorems possible involving the above-mentioned five types of transformations. Every other combination has been shown by an example to lead to a false theorem.

Let $T_1(A) = B$ and $T_2(B) = C$. Then by $T = T_2 \cdot T_1$ we mean $T(A) = T_2[T_1(A)] = T_2(B) = C$.

The two new theorems are as follows:

THEOREM 3.1. *Let the collection $\mathfrak{G}$ be the totality of all single-point sets. If T_1 is monotone and non-separating modulo $\mathfrak{G}$ and if T_2 is non-separating modulo $\mathfrak{G}$, then T is also non-separating modulo $\mathfrak{G}$.*

Proof. Suppose T is not non-separating modulo $\mathfrak{G}$; i.e., assume there is a point x in C such that $T^{-1}(x)$ separates two points a_1 and a_2 in A and no single point of $T^{-1}(x)$ does this. Let $A - T^{-1}(x) = A_1 + A_2$ be a separation, where

A_1 contains a_1 and A_2 contains a_2 . Consider $B_1 = T_1(A_1)$ and $B_2 = T_1(A_2)$. Since T_1 is monotone, $B_1 \cdot B_2 = 0$. Then it follows from the continuity of T_1 and the compactness of A that B_1 and B_2 are separated sets. Hence $B - T_2^{-1}(x) = B_1 + B_2$ is a separation of B , where B_1 contains $T_1(a_1)$ and B_2 contains $T_1(a_2)$. Since T_2 is non-separating modulo $\mathcal{G}$, some single point r of $T_2^{-1}(x)$ must separate $T_1(a_1)$ and $T_1(a_2)$ in B . Thus $B - r = B_3 + B_4$ is a separation, where B_3 contains $T_1(a_1)$ and B_4 contains $T_1(a_2)$. Then $A - T_1^{-1}(r) = T_1^{-1}(B_3) + T_1^{-1}(B_4)$ is a separation of A in which the first set contains a_1 and the second contains a_2 . Since T_1 is non-separating modulo $\mathcal{G}$, some single point y of $T_1^{-1}(r)$ must also separate a_1 and a_2 in A . But then y is contained in $T^{-1}(x)$, and this contradicts our first assumption. Hence T must be non-separating modulo $\mathcal{G}$.

THEOREM 3.2. *Let the collection $\mathcal{G}$ be the totality of all single-point sets. If T_1 is monotone and non-separating modulo $\mathcal{G}$, and if T_2 is non-alternating modulo $\mathcal{G}$, then T is also non-alternating modulo $\mathcal{G}$.*

Proof. Suppose T is not non-alternating modulo $\mathcal{G}$; i.e., assume there are two points x and y in C such that $T^{-1}(x)$ separates two points a_1 and a_2 of $T^{-1}(y)$ in A , and no single point of $T^{-1}(x)$ does this. Let $A - T^{-1}(x) = A_1 + A_2$ be a separation of A , where A_1 contains a_1 and A_2 contains a_2 . Consider $B_1 = T_1(A_1)$ and $B_2 = T_1(A_2)$. As in the preceding theorem, B_1 and B_2 are separated sets, and $B - T_2^{-1}(x) = B_1 + B_2$, where B_1 contains $T_1(a_1)$ and B_2 contains $T_1(a_2)$. The sets $T_1(a_1)$ and $T_1(a_2)$ are contained in $T_2^{-1}(y)$, and since T_2 is non-alternating modulo $\mathcal{G}$, it follows that $T_2^{-1}(x)$ contains some single point r which also separates $T_1(a_1)$ and $T_1(a_2)$ in B . Thus $B - r = B_3 + B_4$ is a separation, where B_3 contains $T_1(a_1)$ and B_4 contains $T_1(a_2)$. Then $A - T_1^{-1}(r) = T_1^{-1}(B_3) + T_1^{-1}(B_4)$ is a separation of A in which the first set contains a_1 and the second contains a_2 . Since T_1 is non-separating modulo $\mathcal{G}$, some single point s of $T_1^{-1}(r)$ must separate a_1 and a_2 in A . But then s is contained in $T^{-1}(x)$, and this contradicts the first assumption. Hence T must be non-alternating modulo $\mathcal{G}$.

The problem of generalizing these product and factor theorems so that they would hold for more general collections $\mathcal{G}$ was considered, and we found that in most cases such severe restrictions had to be imposed that the theorems obtained were of no importance or interest. However, two theorems of Vance's are generalizable with only slight restrictions, and they give the following theorems which hold for collections $\mathcal{G}$ unrestricted except for the condition stated in the hypothesis.

THEOREM 3.3. *If the collection $\mathcal{G}$ is closed under continuous transformations² and if T is non-separating modulo $\mathcal{G}$ and T_1 is monotone, then T_2 is non-separating modulo $\mathcal{G}$.³*

² **DEFINITION.** The collection $\mathcal{G}$ is said to be closed under continuous transformations if the continuous transform of any set in the collection is also a set of the collection.

³ This theorem may be stated in a somewhat more general fashion as follows: If T is non-separating modulo $\mathcal{G}$ and T_1 is monotone, then T_2 is non-separating modulo $T_1(\mathcal{G})$, where $T_1(\mathcal{G})$ is the collection of sets in B which are images under T_1 of those sets in A which belong to $\mathcal{G}$. The same proof is valid for either statement of the theorem.

Proof. Assume T_2 is not non-separating modulo $\mathcal{G}$; i.e., assume there is a point x of C such that $T_2^{-1}(x)$ separates two points b_1 and b_2 of B and $T_2^{-1}(x)$ contains no set of the collection $\mathcal{G}$ which does this. Let $B - T_2^{-1}(x) = B_1 + B_2$ be a separation of B , where B_1 contains b_1 and B_2 contains b_2 . Then $A - T^{-1}(x) = T_1^{-1}(B_1) + T_1^{-1}(B_2)$ is a separation of A , in which $T_1^{-1}(B_1)$ contains $T_1^{-1}(b_1)$, and $T_1^{-1}(B_2)$ contains $T_1^{-1}(b_2)$. Let a_1 be a point of $T_1^{-1}(b_1)$ and a_2 be a point of $T_1^{-1}(b_2)$. Then, since T is non-separating modulo $\mathcal{G}$, the set $T^{-1}(x)$ contains a set K of the collection $\mathcal{G}$ which separates a_1 and a_2 in A . Let $A - K = A_1 + A_2$ be a separation, where A_1 contains a_1 and A_2 contains a_2 . Consider the sets $[T_1(A_1) - T_1(K)]$ and $[T_1(A_2) - T_1(K)]$. The first set in brackets contains the point b_1 because the point a_1 of $T_1^{-1}(b_1)$ is contained in A_1 , and $T_1(K)$ is a subset of $T_2^{-1}(x)$, which does not contain b_1 . Likewise, the second set in brackets contains the point b_2 . Since T_1 is monotone, these two sets are disjoint; and then because of the compactness of A and the continuity of T_1 , it follows that they are separated sets. Thus we have $B - T_1(K) = [T_1(A_1) - T_1(K)] + [T_1(A_2) - T_1(K)]$ is a separation of B . The set K is a set of the collection $\mathcal{G}$ and therefore, by hypothesis, $T_1(K)$ is also a set of the collection $\mathcal{G}$. But then the separation of B by $T_1(K)$ contradicts our first assumption. Hence T_2 must be non-separating modulo $\mathcal{G}$.

THEOREM 3.4. *If the collection $\mathcal{G}$ is closed under continuous transformations, and if T is non-alternating modulo $\mathcal{G}$ and T_1 is monotone, then T_2 is non-alternating modulo $\mathcal{G}$.⁴*

Proof. Assume T_2 is not non-alternating modulo $\mathcal{G}$; i.e., assume there are points x and y of C such that $T_2^{-1}(x)$ separates two points b_1 and b_2 of $T_2^{-1}(y)$ in B , and $T_2^{-1}(x)$ contains no set of the collection $\mathcal{G}$ which does this. Let $B - T_2^{-1}(x) = B_1 + B_2$ be a separation, where B_1 contains b_1 and B_2 contains b_2 . Then $A - T^{-1}(x) = T_1^{-1}(B_1) + T_1^{-1}(B_2)$ is a separation of A in which $T_1^{-1}(B_1)$ contains $T_1^{-1}(b_1)$ and $T_1^{-1}(B_2)$ contains $T_1^{-1}(b_2)$. Let a_1 be a point of $T_1^{-1}(b_1)$ and a_2 be a point of $T_1^{-1}(b_2)$. Then a_1 and a_2 are points of $T^{-1}(y)$, and since T is non-alternating modulo $\mathcal{G}$, the set $T^{-1}(y)$ contains a set K of the collection $\mathcal{G}$ which separates a_1 and a_2 in A . Let $A - K = A_1 + A_2$ be a separation of A , where A_1 contains a_1 and A_2 contains a_2 . Then, just as in the preceding theorem, we have $B - T_1(K) = [T_1(A_1) - T_1(K)] + [T_1(A_2) - T_1(K)]$ is a separation of B . The first set in brackets contains the point b_1 because the point a_1 of $T_1^{-1}(b_1)$ is contained in A_1 , and $T_1(K)$ is a subset of $T_2^{-1}(x)$, which does not contain b_1 . Likewise, the second set in brackets contains the point b_2 . The set K is a set of the collection $\mathcal{G}$ and therefore, by hypothesis, $T_1(K)$ is also a set of the collection $\mathcal{G}$. But then the separation of B by $T_1(K)$ contradicts our first assumption. Hence T_2 must be non-alternating modulo $\mathcal{G}$.

The condition placed in the hypotheses of the two preceding theorems, that

⁴ This theorem may also be stated in a more general fashion: *If T is non-alternating modulo $\mathcal{G}$ and T_1 is monotone, then T_2 is non-alternating modulo $T_1(\mathcal{G})$.*

the continuous transformation of a set of the collection $\mathcal{G}$ is again a set of the collection $\mathcal{G}$, seems to be a rather weak one, and hence it does not restrict seriously the applications of the theorem. The condition is satisfied by all the collections $\mathcal{G}$ which we have had in mind; i.e., collections consisting of single points, finite point sets, connected sets, and sets having a finite number of components.

4. Applications involving special collections $\mathcal{G}$. In this section various particular collections $\mathcal{G}$ will be considered, and the behavior of the transformations modulo $\mathcal{G}$ for these definitions of $\mathcal{G}$ will be studied on certain special curves and surfaces.

THEOREM 4.1. *If A is an n -coherent Peano continuum,⁵ then any continuous transformation T on A is completely non-separating modulo $\mathcal{G}$, where the collection $\mathcal{G}$ is the family of all sets consisting of n or fewer components.*

Proof. Suppose the continuous transformation T is not completely non-separating modulo $\mathcal{G}$; i.e., assume there is a point x of B and a closed set M contained in $T^{-1}(x)$ such that M separates two points a_1 and a_2 in A and M contains no set of the collection $\mathcal{G}$ which does this. Let $A - M = A_1 + A_2$ be a separation, where A_1 contains a_1 and A_2 contains a_2 . Denote by C the component of A_1 containing a_1 . Then a_2 is contained in $A - \bar{C}$. Let D be the component of $A - \bar{C}$ containing a_2 . The sets C and D are thus both open sets, since in Peano space components of open sets are open. The set $A - D$ is connected (see [1]). Then $\overline{A - D}$ and $\bar{D}$ are continua. Obviously $A = \bar{D} + \overline{A - D}$. Therefore, since A is n -coherent, $\bar{D} \cdot \overline{A - D}$ consists of n or fewer components, i.e., $\bar{D} \cdot \overline{A - D}$ is a set of the collection $\mathcal{G}$.

Let $F = \bar{D} \cdot \overline{A - D}$. Then F is the frontier of D . Further, F is contained in $\bar{D} - D$, because no point of the open set D can be a limit point of its complement $A - D$. Also $\bar{D} - D$ is contained in $\bar{C}$, since D is a component of $A - \bar{C}$, and any limit point of D which was not in $\bar{C}$ would have been added to D . Furthermore, since C is an open set and D is in $A - C$, the set C contains no limit points of D . Hence $\bar{D} - D$ is contained in $\bar{C} - C$. Also $\bar{C} - C$ is contained in M , as the following reasoning shows. Consider any limit point y of C . The point y is not an element of A_2 , since C is in A_1 which is separated from A_2 . If y is an element of A_1 , then it would be contained in the component C . Thus the only other possibility is that y is an element of M . Bringing all these facts together we have:

$$F = \bar{D} \cdot \overline{A - D} \subset \bar{D} - D \subset \bar{C} - C \subset M.$$

Now $A - F = (\bar{D} + \overline{A - D}) - F = (\bar{D} - F) + (\overline{A - D} - F)$. These two sets in parentheses are separated, because when the common part of two

⁵ DEFINITION. The continuum A is said to be n -coherent if, whenever A is expressed as the sum of two continua, $A = A_1 + A_2$, then $A_1 \cdot A_2$ consists of n or fewer components. In the case in which $n = 1$, A is called *unicoherent*.

continua is subtracted out, neither remainder can contain a limit point of the other. Now $\bar{D} - F$ contains a_2 , since a_2 is in the interior of D , and F is contained in $\bar{D} - D$. Also, $\overline{A - D} - F$ contains a_1 , because D is contained in $A - \bar{C}$, hence $\bar{C} \subset A - D \subset \overline{A - D}$; and a_1 is an element of C , and F is contained in $\bar{C} - C$.

Thus we have contained in M a subset F of the collection $\mathcal{G}$ which separates a_1 and a_2 in A . This is a contradiction. Hence T must be completely non-separating modulo $\mathcal{G}$.

This theorem is not true in non-Peanian spaces, as the following example shows for the case when $n = 1$. Let A consist of all points in the analytic plane satisfying one of the following conditions:

(a) $x = 0, 0 \leq y \leq 1$;

(b) $y = \frac{1}{i}, 0 \leq x \leq 1, i = 1, \dots, n, \dots$;

(c) $y = 0, 0 \leq x \leq 1$.

A is then a unicoherent continuum. Define the transformation T to be one which sends all points of A which have an abscissa equal to $\frac{1}{2}$ into the point $(\frac{1}{2}, 0)$ and let T be a homeomorphism elsewhere. This transformation is continuous. But the inverse of the point $(\frac{1}{2}, 0)$ separates the two points $(1, 0)$ and $(0, 0)$ in A , and no single component of the inverse effects this separation. Hence T is not non-separating modulo $\mathcal{G}$, where the collection $\mathcal{G}$ is the family of all sets consisting of one component, i.e., connected sets.

THEOREM 4.2. *Let the collection $\mathcal{G}$ be any family of closed subsets of A which includes all single-point sets. A necessary and sufficient condition that no set of the collection $\mathcal{G}$ separate the space A is that every transformation on A which is non-separating modulo $\mathcal{G}$ be also non-separating in the strict sense.*

Proof. Necessity. Suppose A has the property that no set of the collection $\mathcal{G}$ separates A , but assume that there is a transformation T which is non-separating modulo $\mathcal{G}$ but which is not non-separating in the strict sense. This means that there is a point x of B such that $T^{-1}(x)$ separates two points a_1 and a_2 in A , and $T^{-1}(x)$ contains a set of the collection $\mathcal{G}$ which also separates a_1 and a_2 in A . But this contradicts the hypothesis that no set of the collection $\mathcal{G}$ separates A . Hence T must be non-separating in the strict sense.

Sufficiency. Suppose that any transformation on A which is non-separating modulo $\mathcal{G}$ is also non-separating in the strict sense. Now assume that some set M of the collection $\mathcal{G}$ separates A . Then, since M is a closed set, the decomposition of A which has M as one element and all other points in A individually as elements is upper semi-continuous [4], and hence this decomposition defines a continuous transformation T on A which sends M into one point x and is a homeomorphism elsewhere on A (cf. [2]). Now every inverse set $T^{-1}(y)$ belongs to the collection $\mathcal{G}$, for $T^{-1}(y)$ is either a single point of A or the set M and hence belongs to $\mathcal{G}$ in either case. Hence every inverse set $T^{-1}(y)$ which separates A contains a separating subset (namely, itself) belonging to $\mathcal{G}$, and thus T

is non-separating modulo $\mathcal{G}$. But T is not non-separating in the strict sense, for $T^{-1}(x) = M$ separates A . This contradicts our hypothesis and hence A must have the property that no set of the collection $\mathcal{G}$ separates A .

The following theorems are stated without proof, because the proofs are very similar to the one given for Theorem 4.2.

THEOREM 4.3. *Let $\mathcal{G}$ be any family of closed subsets of A which includes all subsets consisting of one or two points. In order that no set of the collection $\mathcal{G}$ separate A it is necessary and sufficient that every transformation on A which is non-alternating modulo $\mathcal{G}$ be also non-alternating in the strict sense.*

THEOREM 4.4. *Let $\mathcal{G}$ be any hereditary family⁶ of closed subsets of A which includes all single-point subsets. In order that no set of the family $\mathcal{G}$ separate A it is necessary and sufficient that every transformation on A which is completely non-separating modulo $\mathcal{G}$ be also completely non-separating in the strict sense.*

THEOREM 4.5. *Let $\mathcal{G}$ be any hereditary family of closed subsets of A which includes all sets consisting of one or two points. In order that no set of the family $\mathcal{G}$ separate A it is necessary and sufficient that every transformation on A which is completely non-alternating modulo $\mathcal{G}$ be also completely non-alternating in the strict sense.*

A number of corollaries to the preceding theorems are easily deduced, and a few of them will be stated here. These also are given without proof, for either they are just special cases of the preceding theorems, or their proofs are very similar to the proof of Theorem 4.2.

COROLLARY. *A necessary and sufficient condition that a Peano continuum A be a simple closed curve is that any transformation on A which is* $\left\{ \begin{array}{l} \text{non-separating} \\ \text{non-alternating} \end{array} \right\}$ *modulo $\mathcal{G}$, where $\mathcal{G}$ is the collection of all continua in A , be also* $\left\{ \begin{array}{l} \text{non-separating} \\ \text{non-alternating} \end{array} \right\}$ *in the strict sense.*

This corollary makes use of the theorem that a Peano continuum is a simple closed curve if and only if it is not separated by any subcontinuum (cf. [3]).

COROLLARY. *Let the collection $\mathcal{G}$ consist of all single-point sets in A . In order that the space A have no cut point it is necessary and sufficient that every*

transformation T on A which is $\left\{ \begin{array}{l} \text{completely non-separating} \\ \text{completely non-alternating} \\ \text{non-separating} \\ \text{non-alternating} \end{array} \right\}$ *modulo $\mathcal{G}$ be*

also $\left\{ \begin{array}{l} \text{completely non-separating} \\ \text{completely non-alternating} \\ \text{non-separating} \\ \text{non-alternating} \end{array} \right\}$ *in the strict sense.*

⁶ **DEFINITION.** A family $\mathcal{G}$ of closed sets is said to be hereditary if every closed subset of any set of the family $\mathcal{G}$ also belongs to $\mathcal{G}$.

5. Locally non-separating transformations. Locally non-separating transformations were defined by Vance [5] as follows:

A continuous transformation T is called *locally non-separating at a point* p of A provided for any neighborhood U_p of p there exists a neighborhood V_p of p , contained in U_p , such that if u and v are two points of $V_p - T^{-1}[T(p)]$, then u and v lie in a connected subset of $U_p - T^{-1}[T(p)]$. A continuous transformation is called *locally non-separating on the space* A if it is locally non-separating at every point of A .

Vance gave examples to show that non-separating transformations need not be locally non-separating, and that locally non-separating transformations need not be non-separating.

THEOREM 5.1. *If A contains an open set U such that (a) $\bar{U}$ is a Peano continuum, (b) every non-end-point of $\bar{U}$ is a local cut point⁷ of $\bar{U}$, and (c) $\bar{U} - U$ is at most countable, and if T is locally non-separating at all points of $\bar{U}$, then $T(\bar{U})$ is a point.*

Proof. Assume $T(\bar{U})$ contains at least two points, y_1 and y_2 . Consider any two points of $\bar{U}$, x_1 and x_2 , such that $T(x_1) = y_1$ and $T(x_2) = y_2$. Then, since $\bar{U}$ is Peanian, $\bar{U}$ contains an arc α from x_1 to x_2 . There are uncountably many points in $U \cdot \alpha$, because α is uncountable and $\bar{U} - U$ is countable. At most two points of $U \cdot \alpha$ can be end points of $\bar{U}$: hence $U \cdot \alpha$ contains uncountably many local cut points of $\bar{U}$. No point of $U \cdot \alpha$ which is a local cut point of $\bar{U}$ can be a component of any set of the form $\bar{U} \cdot T^{-1}(x)$, because if it were, then T would not be locally non-separating at that point.

It is not possible for all the points of $U \cdot \alpha$ which are local cut points of $\bar{U}$ to lie in one set $T^{-1}(x)$, for any closed set containing all such points contains all points of α , and then we would have $T(x_1) = T(x_2)$. This is a contradiction to our original assumption.

We now have that every point of $U \cdot \alpha$ which is a local cut point of $\bar{U}$ must lie in a non-degenerate component of a set of the form $\bar{U} \cdot T^{-1}(x)$, and not all such points can lie in one set $T^{-1}(x)$. The set $T(\alpha)$ is a connected set containing more than one point, and hence it contains uncountably many points. Therefore α has points in uncountably many sets of the type $\bar{U} \cdot T^{-1}(x)$, and all but a countable number of these points are points of U which are local cut points of $\bar{U}$. Therefore there are uncountably many mutually exclusive non-degenerate components of different sets $\bar{U} \cdot T^{-1}(x)$, each component having at least one point in common with $U \cdot \alpha$.

Of this set of components just described, only a countable number can be contained in the arc α . Therefore uncountably many of them must contain points whose distance from α is greater than some $\epsilon > 0$. Then it follows readily that in each of these components there is an arc which contains a point

⁷ DEFINITION. A point p of a continuum M is called a *local cut point* of M provided there is a neighborhood U of p such that p is a cut point of the component of $\bar{U} \cdot M$ which contains p .

whose distance from α is greater than ϵ and which has exactly one point in common with α . The set M of points of intersection of these arcs with α is uncountable, and hence M contains a set of condensation points. Then there exists in M a point p_0 which is the limit point of a convergent sequence $\{p_i\}$ of points of M . Let α_i denote the arc which contains p_i . Then $\{\alpha_i\}$ ($i = 1, 2, \dots$) is a sequence of sets which converges to a continuum C_0 containing p_0 . The decomposition of $\bar{U}$ into sets of the form $\bar{U} \cdot T^{-1}(x)$ is upper semi-continuous and since each set α_i is contained in a different set $\bar{U} \cdot T^{-1}(x)$, it follows that C_0 is a subcontinuum of the set $\bar{U} \cdot T^{-1}(x)$ which contains p_0 . Then $C_0 \cdot \alpha_i = 0$ for $i = 1, 2, \dots$. Furthermore, C_0 contains points whose distance from α is equal to or greater than ϵ , since each set α_i contains such points. Let A denote the minimal subarc of α containing all points p_i . Then let $K = A + C_0 + \sum_{i=1}^{\infty} \alpha_i$. The set K is connected and closed, and hence is a continuum. However, K is non-Peanian because it fails to be locally connected at any point of $C_0 - A$. This follows from the fact that in order to join a point of α_i to a point of $C_0 - A$ by a connected set in K it is necessary to pass through the point p_i in A . Thus we have in $\bar{U}$ a subcontinuum K which is non-Peanian. But it follows from the hypotheses of the theorem that $\bar{U}$ is hereditarily locally connected (cf. [8]). This is a contradiction. Hence it follows that $T(\bar{U})$ must be a single point, and the theorem is proved.

COROLLARY 5.11. *If A is a dendrite or finite graph and T is locally non-separating, then $T(A)$ is a point.*

COROLLARY 5.12. *If T is locally non-separating and A contains a connected open set U such that $\bar{U}$ is a finite graph, then $T(\bar{U})$ is a point.*

COROLLARY 5.13. *If every cyclic element of the Peano continuum A is a finite graph, and T is locally non-separating, then $T(A)$ is a point.*

The condition that $\bar{U} - U$ shall be at most countable, in the hypothesis of Theorem 5.1, is necessary, as the following example demonstrates: We start with a square plus its interior. On one edge of the square designate by K a Cantor non-dense perfect set. Then, by using the complementary intervals of this Cantor set as diameters, construct semi-circular loops outside the square. Denote by $\bar{U}$ the Cantor set K plus the semi-circular loops. Let $U = \bar{U} - K$. Then $\bar{U} - U = K$ is uncountably infinite. Decompose the entire set A (square plus loops) as follows: Let each loop plus the complementary interval on which it was constructed be an element and every other point be individually an element. This decomposition is upper semi-continuous and thus induces a continuous transformation T carrying each element into a point of a space B . This transformation is easily seen to be locally non-separating on the entire space A . Further, $\bar{U}$ satisfies all the conditions of Theorem 5.1 except that $\bar{U} - U$ is not countable. But $T(\bar{U})$ is not a point: in fact it is an arc of the space B . Hence the condition that $\bar{U} - U$ be countable is necessary.

6. Locally non-alternating transformations. The definition of a locally non-alternating transformation, as given by Vance, is as follows: A continuous transformation is called *locally non-alternating at a point* p of $T^{-1}(x)$ provided for any neighborhood U_p of p , there exists a neighborhood V_p lying in U_p such that if y is any point of B distinct from x , and u and v are any two points of $T^{-1}(y) \cdot V_p$, then u and v lie in a connected subset of $U_p - T^{-1}(x)$. A continuous transformation is called *locally non-alternating on a space* A if it is locally non-alternating at all points of A .

Vance gave simple examples to show that a non-alternating transformation is not necessarily locally non-alternating, and that a locally non-alternating transformation is not necessarily non-alternating.

THEOREM 6.1. *If T is a light⁸ locally non-alternating transformation from the dendrite A to the dendrite B , then T is a homeomorphism.*

Proof. Assume T is not a homeomorphism. Then some two points a_0 and b_0 of A must go into the same point x_0 of B . Consider the arc $a_0b_0 = \alpha_0$. It must contain a point p such that $T(p) = q \neq x_0$, for otherwise $T^{-1}(x_0)$ would contain the connected set α_0 and hence T would not be light. Also, there must be some point x_1 of B , distinct from q and x_0 , such that $T^{-1}(x_1) \cdot \text{arc } a_0p \neq 0$ and $T^{-1}(x_1) \cdot \text{arc } pb_0 \neq 0$, for if there were no such point x_1 , then we would have $T(\text{arc } a_0p) \cdot T(\text{arc } pb_0) = q + x_0$. Thus there would be two connected sets running from q to x_0 and having only $q + x_0$ in common, but this is impossible in a dendrite. Let a_1 denote the first point of $T^{-1}(x_1)$ in the order from p to a_0 ; b_1 the first point of $T^{-1}(x_1)$ in the order from p to b_0 . Then arc $a_1b_1 = \alpha_1$ lies entirely in α_0 . By repeating the above argument we can obtain an arc α_2 lying within α_1 , an arc α_3 within α_2 , etc. Repeat the process indefinitely.

Let $\alpha_\omega = \bigcap_{i=1}^{\infty} \alpha_i$. If α_ω is a point, then T is not locally non-alternating at this point because $a_i \rightarrow \alpha_\omega$, $b_i \rightarrow \alpha_\omega$ and $T(a_i) = T(b_i)$, but a_i and b_i cannot be connected inside small neighborhoods of α_ω except through α_ω . Since α_ω is not a point, it must be an arc with end points a_ω and b_ω , and it follows from the continuity of T that $T(a_\omega) = T(b_\omega)$. Then the arc $\alpha_{\omega+1}$ can be obtained just as the arcs α_i ($i < \omega$) were obtained. This process may be continued indefinitely and arcs α_β defined for all ordinals of the first and second classes, unless for some limiting ordinal β , α_β is a point. But this is impossible for at this point T would fail to be locally non-alternating. But the process cannot continue through all the ordinals of the first and second classes, for then A would contain an uncountable monotonic decreasing sequence of continua, and this is impossible.

Hence T must be one-to-one, and therefore a homeomorphism.

⁸ DEFINITION. A transformation T is said to be light if every set $T^{-1}(x)$ is totally disconnected.

The following theorems proved by Vance follow as corollaries of this theorem:

COROLLARY 6.11. *If T is locally non-alternating, with A an arc and B a dendrite, and if $T^{-1}(x)$ contains no open set, then T is a homeomorphism and thus B is an arc.*

COROLLARY 6.12. *If T is locally non-alternating where B is a dendrite and A is a dendrite in which the branch points are dense on no arc of A , and if $T^{-1}(x)$ contains no open set, then T is a homeomorphism.*

The remainder of this section will be devoted to consideration of the relationship between locally non-alternating transformations and 0-regular transformations.⁹

THEOREM 6.2. *If the transformation $T(A) = B$ is 0-regular, then T is locally non-alternating on A .*

Proof. This theorem follows readily with the help of the following theorem of Wallace [6]: If A is a compact continuum, a necessary and sufficient condition that the interior transformation $T(A) = B$ shall be 0-regular is that for each $\epsilon > 0$ there exist a $\delta > 0$ such that if u and v are in A with $\rho(u, v) < \delta$ and $T(u) = T(v)$, then u and v lie in an ϵ -continuum in $T^{-1}[T(u)] = T^{-1}[T(v)]$. Let p be any point of A , and let U_p be any neighborhood of p . Then, since we are dealing with a metric space, U_p contains a spherical neighborhood S_p with center at p and radius equal to k , for some k . Let $\epsilon = \frac{1}{2}k$. Then, for this ϵ , there is determined a number $\delta > 0$ which satisfies the condition given in the theorem of Wallace just quoted. Let V_p be a spherical neighborhood with center at p and radius equal to $\frac{1}{2}\delta$. Now consider any point y distinct from $T(p)$ in B , and let u and v be any two points of $T^{-1}(y) \cdot V_p$. Then $\rho(u, v) < \delta$, and hence u and v lie in an ϵ -continuum C in $T^{-1}(y)$, by Wallace's theorem. Because $\epsilon = \frac{1}{2}k$, C is contained in S_p and hence in U_p . Furthermore, since $y \neq T(p)$ and C is contained in $T^{-1}(y)$, $C \cdot T^{-1}[T(p)] = \emptyset$. Hence u and v , any two points of $T^{-1}(y) \cdot V_p$, lie in a connected subset of $U_p - T^{-1}[T(p)]$. Therefore, by definition, T is locally non-alternating at the point p . Since p was any point of A , T is locally non-alternating on A .

The converse of Theorem 6.2 is not true in general, i.e., an interior, locally non-alternating transformation is not necessarily 0-regular. The following example demonstrates this remark: Let A denote a spherical shell of outer radius unity and inner radius $\frac{1}{2}$. Let T be the orthogonal projection of A onto a tangent plane. This transformation is interior and locally non-alternating

⁹ **DEFINITION.** (1) If the sequence of closed sets M_n converges to M , then $M_n \rightarrow M$ 0-regularly provided that for each $\epsilon > 0$ there are positive numbers δ and N such that for $n > N$ any two points x and y in M_n with $\rho(x, y) < \delta$ lie in a continuum in M_n of diameter less than ϵ .

(2) A continuous transformation T is called 0-regular provided that if $y_n \rightarrow y$ in B , the sets $T^{-1}(y_n) \rightarrow T^{-1}(y)$ 0-regularly in A .

at all points of A , but it fails to be 0-regular at any point of the inner surface of A whose distance from the tangent plane is unity.

By looking carefully at the requirements of 0-regular and locally non-alternating transformations it is easy to see why 0-regularity is the stronger type of transformation. In non-rigorous terms, 0-regularity requires that if u and v , two points of a set $T^{-1}(y)$, are sufficiently close together, they must lie in a small connected set C which is also contained in $T^{-1}(y)$. Locally non-alternating merely requires that two such points u and v shall lie in a small connected set C , and does not require that C shall be contained in $T^{-1}(y)$, but only that it shall avoid $T^{-1}(x)$ for a specified point x . This is obviously a weaker condition, and so the locally non-alternating condition is not sufficient in general to imply 0-regularity.

THEOREM 6.3. *On a dendrite, 0-regular transformations and interior, locally non-alternating transformations are equivalent, since both are homeomorphisms.*

Proof. It was proved by Wallace [6] that any 0-regular transformation on a dendrite is a homeomorphism. G. T. Whyburn showed that on a dendrite any interior transformation is also light (cf. [11]), and that the image of a dendrite under an interior transformation is again a dendrite cf. [10]. Then by Theorem 6.1 at the beginning of this section, it follows that an interior locally non-alternating transformation on a dendrite is a homeomorphism. Hence the theorem follows.

BIBLIOGRAPHY

1. B. KNASTER AND C. KURATOWSKI, *Sur les ensembles connexes*, Fundamenta Mathematicae, vol. 2(1921), p. 214.
2. C. KURATOWSKI, *Les décompositions semi-continues*, Fundamenta Mathematicae, vol. 11(1928), pp. 169-185.
3. C. KURATOWSKI, *Sur les continus de Jordan*, Fundamenta Mathematicae, vol. 5(1924), p. 119.
4. R. L. MOORE, *Concerning upper semi-continuous collections of continua*, Transactions of the American Mathematical Society, vol. 27(1925), pp. 416-428.
5. E. P. VANCE, *Generalizations of non-alternating and non-separating transformations*, this Journal, vol. 6(1940), pp. 66-79.
6. A. D. WALLACE, *On 0-regular transformations*, American Journal of Mathematics, vol. 62(1940), pp. 277-284.
7. J. F. WARDWELL, *Non-separating transformations*, this Journal, vol. 2(1936), pp. 745-750.
8. G. T. WHYBURN, *Concerning points of continuous curves defined by certain im kleinen properties*, Mathematische Annalen, vol. 102(1930), p. 321, Theorem 11.
9. G. T. WHYBURN, *Non-alternating transformations*, American Journal of Mathematics, vol. 56(1934), pp. 294-302.
10. G. T. WHYBURN, *Interior transformations on compact sets*, this Journal, vol. 3(1937), pp. 370-381.
11. G. T. WHYBURN, *Interior transformations on certain curves*, this Journal, vol. 4(1938), pp. 607-612.

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